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A STUDY ON GENERALIZED HERMITE POLYNOMIALS

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Abstract

In this paper, we obtain generating functions involving hyper geometric functions. Rodrigues type formula of Hermite polynomials which is closely related to generalized Hermite polynomials of Dattoli et. al. These results provide useful extensions of the well known results of classical Hermite polynomials $H_n(x)$.

***Keywords**: Generating functions, Hermite polynomials, Gould-Hopper polynomials, Rodrigues type formula,Dattoli et.al.*

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Introduction: The Gould-Hopper polynomials $g_n^m(x,y)$ [2; p.512 (18)] see also [3; p.58 (6.2)] are generalization of classical Hermite polynomials $H_n(x)$ [6; p.187(2)]. The notation $H_n^{(m)}(x, y)$ for $g_n^{(m)}(x, y)$ was givenby Dattoli et.al. and by Pathan, Yasmeen and Qureshi [4]. It is defined by [3; 6; p.76(6)]

$$g_n^m(x,y) = H_n^{(m)}(x,y) = \sum_{k=0}^{[n/m]} \frac{n!\, y^k\, x^{n-mk}}{k!\,(n-mk)!}$$

…(1.1)

$$= x^n \; {}_mF_0\left[\begin{matrix}\Delta(m;-n; \\ - \end{matrix}\;;\; y\left(\frac{-m}{x}\right)^m\right]. \qquad \textbf{… (1.2)}$$

where m is a positive integer and Δ (m; -n) abbreviates the array of m parameters,

$$\frac{-n}{m}, \frac{-n+1}{m}, \ldots, \frac{-n+m-1}{m}; \; m \geq 1.$$

These polynomials reduce, when m = 2 and y = −1, to the classical Hermite polynomials. The equation (1.1) can be derived from the following generating relations [2; p.512 (19)]

$$\sum_{n=0}^{\infty} H_n^{(m)}(x, y)\frac{t^n}{n!} = \exp\,(xt + yt^m) \qquad \textbf{… (1.3)}$$

and reduces to Hermite polynomials of two variables [2; p.511(11)]

$$H_n(x,y) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \; n!(2x)^{n-2k} y^k}{k!(n-2k)!} \quad \textbf{... (1.4)}$$

$$= H_n^{(2)} \; (2x,-y) \quad \textbf{... (1.5)}$$

which can be derived from the generating relations [2; p.510(8)]

$$\sum_{n=0}^{\infty} \frac{H_n(x,y)t^n}{n!} = \exp\;(2xt - yt^2) \quad \textbf{... (1.6)}$$

$$H_n\;(x,1) = H_n(x) \quad \textbf{... (1.7)}$$

In this paper we shall give some basic relations and properties involving the generalized Hermite polynomials $H_n(x,y)$ and then take up generating function and Rodrigues type formula for H_n (x, y) and H_n (x) are derived as special cases.

Generating Functions

Theorem – 1

Any values of parameters and variables leading to result which do not make sense are tactily excluded then

$$\sum_{n=0}^{\infty} (c)_n H_n(x,\, y)\frac{t^n}{n!} = (1-2xt)^{-c} \times$$

$${}_2F_0\left[\begin{matrix}\frac{c}{2}, \frac{c+1}{2}; \\ - \quad ;\end{matrix} \frac{-4yt^2}{(1-2xt)^2}\right]; |2xt| < 1 \quad \textbf{...(2.1)}$$

Proof: Consider the following series

$$\sum_{n=0}^{\infty} (c)_n H_n(x,\, y)\frac{t^n}{n!} =$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(c)_n \; (-1)^k \; (2x)^{n-2k} \; y^k t^n}{k!\,(n-2k)!}$$

Replacing n by n+2k, and using Legendre's duplication formula, we get

$$= \sum_{k=0}^{\infty} (1-2xt)^{-c-2k} \frac{\left(\frac{c}{2}\right)_k \left(\frac{c+1}{2}\right)_k (-4yt^2)^k}{k!}$$

$$= (1-2xt)^{-c} \; {}_2F_0\left[\begin{matrix}\frac{c}{2}, \frac{c+1}{2} \quad ; \\ - \quad ;\end{matrix} \frac{-4yt^2}{(1-2xt)^2}\right]$$

which is required result (2.1).

Theorem-2

Any values of parameters and variables leading to result which do not make sense are tactily excluded, then

$$\sum_{n=0}^{\infty} {}_2F_0[-n,c;-;v]H_n\ (x,y)\frac{t^n}{n!} = \exp\ (2xt - yt^2)$$

$x[1+2(x-yt)vt]^{-c}{}_2F_0\left[\frac{c}{2},\frac{c+1}{2};-;\frac{-4yv^2t^2}{(1+2xvt-2yvt^2)^2}\right]$ **Proof:** Consider the following series

$$S = \sum_{k=0}^{\infty}\frac{(c)_k H_k(x-yt,y)(-vt)^k}{k!} \qquad \textbf{...(2.3)}$$

Now using [4; p.452 (2.4)], we get

$$S = \exp\ (-2xt + yt^2)\times$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty]}\frac{(-1)^k\ (c)_k\ v^k H_{n+k}\ (x,y)t^{n+k}}{k!\,n!}.$$

Replacing n by n-k and using

$$(-n)_k = \frac{(-1)^k\ n!}{(n-k)!}, 0\le k\le n, \text{we get}$$

$$S = \exp\ (-2xt + yt^2)\sum_{n=0}^{\infty}\ {}_2F_0\ [-n,c;-;v]\ H_n\ (x,y)\frac{t^n}{n!}. \qquad \textbf{... (2.4)}$$

Now again by (2.3) and using (2.1), we get

$$S = [1+2v(x-yt)t]^{-c}\ {}_2F_0\left[\frac{c}{2},\frac{c+1}{2};-;\frac{-4y\,v^2\ t^2}{(1+2xvt-2yvt^2)^2}\right] \qquad \textbf{... (2.5)}$$

Now equating the equation (2.4) and (2.5), we get required result (2.2).

Rodrigues type formula

Theorem-3

If $D = \frac{\partial}{\partial x}$, then H_n (x,y) = 2^n exp $\left(-\frac{y}{4}D^2\right)x^n$

... (2.6)

Proof: Again by generating relations (1.6), we get

$$\sum_{n=0}^{\infty}H_n(x,y)\frac{tn}{n!} = \exp\ (2xt)\sum_{n=0}^{\infty}\frac{(-yt^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-y}{4} D^2 \right)^n \exp\ (2xt)$$

$$= \exp \left(\frac{-y}{4} D^2 \right) \sum_{n=0}^{\infty} \frac{(2xt)^n}{n!}$$

Equating the coefficient of t^n, we get required result (2.6).

Theorem-4

Use the fact that

$\exp\ (2xt - yt^2) = \exp\ [2(xt) - y(xt)^2]\ \exp\ (yt^2\ (x^2\text{-}1))$

… (2.7)

To prove

$$H_n(x, y) = \sum_{s=0}^{[n/2]} \frac{n!\, H_{n-2s}(1, y)\, x^{n-2s} y^s (x^2 - 1)^s}{s!\,(n-2s)!} \qquad \textbf{… (2.8)}$$

Proof: By (2.7) we can write

$$\sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} H_n\ (1, y) \frac{(xt)^n}{n!} \sum_{s=0}^{\infty} \frac{y^s t^{2s}\ (x^2 - 1)^s}{s!}$$

Replacing n by n – 2s, in right hand side and equating the coefficient of t^n, we get required result.

Special Cases

I. For y = 1, then (2.1) reduced to [6; p.190 (1)], which is published by Brafman [1] and for c = 1 was given by Truesdell [8].

For y = 1, v = y, then (2.2) reduces to

$$\sum_{n=0}^{\infty} {}_2F_0[-n, c; -; y] \frac{H_n\ (x)\, t^n}{n!} = \exp\ (2xt - t^2)$$

$$\times [1 + 2y\ (x - t)]^{-c}\, {}_2F_0 \left[\frac{c}{2}, \frac{c+1}{2}; -; \frac{-4y^2 t^2}{(1 + 2xyt - 2yt^2)^2} \right]$$

This is a well known result [6; p.198] and obtained by Brafman [1] with contour integration as the main tool.

II. For y = 1, (2.6) reduce to [5;p.129(2)].

III. For y = 1, then (2.8) reduces to

$$H_n(x) = \sum_{s=0}^{[n/2]} \frac{n!\,H_{n-2s}(1)\,x^{n-2s}(x^2-1)^s}{s!\,(n-2s)!}$$

This is a known result [7; p.133(3)]

Special cases I to III are known formulae of generating functions, Rodrigues type formula for classical Hermite polynomial $H_n(x)$.

REFERENCES

Brafman,F.; Some generating functions of Laguerre and Hermite polynomials. Canad.J.Math. 9, (1957), 180-187.

Dattoli,G., Chiccoli,C., Lorenzutta, S.Torre, A., and Maino, G.; Generalized Bessel Functions and Generalized Hermite polynomials, Jour. Math. Anal. Appl., 178 (1993), 509-516.

Gould, H.W. and Hopper, A.T.; Operational formulas connected with two generalization of Hermite polynomials. Duke Math. J.29, (1962), 51-63.

Pathan, M.A., Yasmeen and Qureshi, M.I.; Linear and bilinear generating function involving Gould-Hopper polynomials, Math. Sce. Res. J. 6 (9) (2002), 449-456.

Pathan, M.A., Kazim, M.A. and Saksena, K.M.; Elements of Special Functions, P.C. Dwadash Shreni and Co. Pvt. Ltd., Aligarh (1972).

Rainville, E.D.; Special Functions. Macmillan, New York; Reprinted by Chelsea Pub. Co., Bronx, New York, 1971.

Saxena, R.K. and Gokhroo, D.C.; Special Functions, J.P.H., Jaipur (1987).

Truesdell, C.; A Unified Theory of Special Function. Princeton Univ. Press, Princeton, New Jersey (1948).